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Draw AC. Then is

$$\begin{aligned} AB^2 &= BC^2+AC^2 = BC^2+CD^2+AD^2-2AD\times CD\cos D \\ &= BC^2+DC^2+AD^2+2AD\times DC\sin CAB \\ &= BC^2+DC^2+AD^2+2\,\frac{AD\times DC\times BC}{AB}. \qquad (1) \end{aligned}$$

In a triangle with the given notation,

$$\frac{1}{a^2}=\frac{\sin^2\frac{1}{2}A}{r^2};\ \frac{1}{b^2}=\frac{\sin^2\frac{1}{2}B}{r^2};\ \frac{1}{c^2}=\frac{\sin^2\frac{1}{2}C}{r^2};$$

$$\begin{aligned} \therefore\quad \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} &= \frac{1}{r^2}\left(\sin^2\tfrac{1}{2}A+\sin^2\tfrac{1}{2}B+\sin^2\tfrac{1}{2}C\right) \\ &= \frac{1}{r^2}\left(1-2\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\right) \\ &= \frac{1}{r^2}\left(1-\frac{2r^3}{abc}\right). \qquad (2) \end{aligned}$$

Substituting values of $\sin\frac{1}{2}A$ &c.,

$$\frac{1}{r^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{2r}{abc}. \qquad (3)$$

Compariug (1) with (3) we find $1\div r^2$ corresponds to AB^2, $1\div a^2$ to BC^2 &c., hence the values of these reciprocals might be made the sides of a trapezium npon AB.

[A demonstration of the abov prop. was also sent by Mr. E. B. Seitz.]

SOLUTIONS OF PROBLEMS IN NUMBER TWO.

SOLUTIONS of problems in No. 2 have been received as follows:

From Amateur, 199; R. J. Adcock, 200, 201; Marcus Baker, 196, 197; Prof. W. P. Casey, 196, 197, 200; Prof. P. E. Chase, 197, 200; Geo. M. Day, 196, 197; E. L. De Forest, 201; Capt. J. L. de Fremery, 196, 197; W. E. Heal, 196; H. Heaton, 196, 197, 198, 200; Prof. E. W. Hyde, 201; Chas. H. Kummell, 198, 201; Prof. J. H. Kershner, 196, 197; W. V. Mc. Knight, 196, 197; Prof. D. J. Mc. Adam, 196, 197, 200; Prof. H. T. J. Ludwick, 198; Artemas Martin, 196, 197, 198; K. S. Putnam, 196, 197; P. Richardson, 197; Prof. J. Scheffer, 196, 197, 199, 200; Prof. T. A. Smith, 196, 197; E. B. Seitz, 196, 197, 198, 200; Anna T. Snyder, 197; T. P. Stowell, 197; C. A. Van Velzer, 196, 197, 200.

196. "The sides of a triangle are respectively x^2+x+1, $2x+1$ and x^2-1, x being any number greater than one; prove that the angle opposite the side x^2+x+1 is equal to 120°."

SOLUTION BY GEO. M. DAY, LOCKPORT, N. Y.

Let y denote the angle opposite the side x^2+x+1, then we have

$$(x^2+x+1)^2 = (2x+1)^2+(x^2-1)^2-2(2x+1)(x^2-1)\cos y.$$

Expanding and reducing, we get

$$\cos y = -\tfrac{1}{2}; \therefore y = 120^\circ.$$

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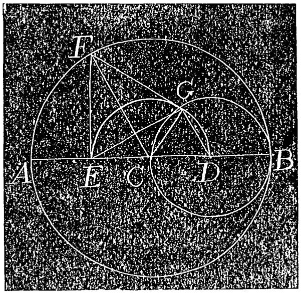
197. "AB is the diameter of a circle whose center is C, D is the middle point of CB, and is the center of a circle whose diameter is CB, and E is the middle p't of AC. EF is perp. to AB and intersects the circle C in F, and EG is tangent to the circle D at G. Prove that EFG is an equilateral triangle."

DEMONSTRATION BY ANNA T. SNYDER, ORANGE, INDIANA.

With center C and radius CD describe the semi-cicle ED. Then, because the angle DGE is a right angle, G is a point in the semi-circle. Draw CG and CF. Then is $CG = CD = DG =$ radius of the circle $D = r$, say; and $CF = CB = ED = 2r =$ radius of the circle C.



Now because $CD = CG = DG = r$, the triangle CDG is equilateral; $\therefore \angle CDG = 60^\circ$, and the complementary angle $DEG = 30^\circ$; therefore $\angle GEF = 60^\circ$.

Because $ED = CF = 2r$, $CE = DG = r$, and the angles at E and G are right angles, $\therefore EG = EF$, and the triangle GEF is isosceles; but it has been shown that the vertical angle $GEF = 60^\circ$, hence the triangle is equiangular, and therefore equilateral.

[In the announcement of this proposition we should have stated that the required demonstration must be independent of trigonometrical functions.]

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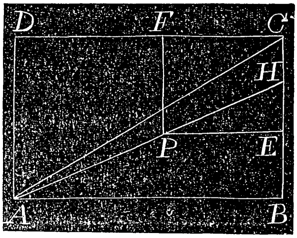
198. "Find the average distance between two points taken at random in the surface of a rectangle, sides a and b."

SOLUTION BY E. B. SEITZ, GREENVILLE OHIO.

From A draw AP to any point P in the triangle ABC.

Now if AP is the distance and the direction of the second point from the first, the area of the rectangle $PECF$ represents the number of ways the two points can be taken.

Let $AB=a$, $BC=b$, $AC=c$, $AP=x$, $AH=x'$, $\angle PAB=\theta$, $\angle CAB=\theta'$, area $PECF=u$, $\Delta=$ the requ'd average, Δ', $\Delta''=$ the average distance between the two points when P is in the triangles ABC, ACD respectively.



Then $x'=a\sec\theta$, $\theta'=\tan^{-1}(b\div a)$, $u=(a-x\cos\theta)(b-x\sin\theta)$; an element of the rectangle at P is $x\,d\theta\,dx$; for Δ' the limits of θ are 0 and θ'; and of x, 0 and x'.

$$\therefore\ \Delta'=\int^{\theta'}\int_0^{x'}ux^2d\theta dx\div\int_0^{\theta'}\int_0^{x'}uxd\theta dx=\frac{8}{a^2b^2}\int_0^{\theta'}\int_0^{x'}ux^2d\theta dx$$

$$=\frac{2a^2}{15b^2}\int_0^{\theta'}(5b-3a\tan\theta)\sec^3\theta d\theta=\frac{c}{3}+\frac{2a^3}{15b^2}-\frac{2c^3}{15b^2}+\frac{a^2}{3b}\log\left(\frac{b+c}{a}\right).$$

By symmetry we have $$\Delta''=\frac{c}{3}+\frac{2b^3}{15a^2}-\frac{2c^3}{15a^2}+\frac{b^2}{3a}\log\left(\frac{a+c}{b}\right).$$

$$\therefore\ \Delta=\tfrac{1}{2}(\Delta'+\Delta'')=\frac{1}{15}\left[\frac{a^3}{b^2}+\frac{b^3}{a^2}+c\left(3-\frac{a^2}{b^2}-\frac{b^2}{a_2}\right)+\frac{5a^2}{2b}\log\left(\frac{b+c}{a}\right)\right.$$
$$\left.+\frac{5b^2}{2a}\log\left(\frac{a+c}{b}\right)\right].$$

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199. "Find an infinite series the sum of which is a square, the sum of n terms of which is a square, and the sum of the remaining terms, after the n terms are taken away, is a square."

SOLUTION BY THE PROPOSER.

Let $S_n^2=$ sum of n terms, and $S_{\infty-n}^2=$ sum of remaining terms. Then

$$S_\infty^2=S_n^2+S_{\infty-n}^2, \tag{1}$$

and the series is

$$S_1^2+(S_2^2-S_1^2)+(S_3^2-S_2^2)+(S_4^2-S_3^2)+\dots=S_\infty^2, \tag{2}$$

of which the general term is $\varphi(n)=S_n^2-S_{n-1}^2$: if its first term follows the same law as the others, it would be

$$\varphi(1)=S_1^2=S_1^2-S_0^2;\ \therefore\ S_0^2=0. \tag{3}$$

In any series

$$1^2=\left(\frac{n^2-r^2}{n^2+r^2}\right)^2+\left(\frac{2rn}{n^2+r^2}\right)^2; \tag{4}$$

$\therefore$ if $S_\infty=1$, $S_n=1$ when n is infinite. In (4) when $n=\infty$,

$$\frac{n^2-r^2}{n^2+r^2}=1,\text{ and }\frac{2rn}{n^2+r^2}=0;\ \therefore\ S_n\text{ must be taken }=\frac{n^2-r^2}{n^2+r^2}.$$

But (3) is not satisfied by this value of S because it vanishes when $n=r$ and not when $n=0$.

In (4) substitute $n+r$ for n, and it becomes

$$1^2=\left(\frac{n^2+2rn}{(n+r)^2+r^2}\right)^2+\left(\frac{2r(n+r)}{(n+r)^2+r^2}\right)^2,$$

in which the first term of the second member vanishes when $n=0$;

$$\therefore S_n=\frac{n^2+2rn}{(n+r)^2+r^2},\ \therefore S_{\infty-n}=\frac{2r(n+r)}{(n+r)^2+r^2}.$$

$$\varphi(n)=S_n^2-S_{n-1}^2=\left(\frac{n^2+2rn}{(n+r)^2+r^2}\right)^2-\left(\frac{(n-1)^2+2r(n-1)}{(n-1+r)^2+r^2}\right)^2.$$

Substitute this in (2) and the required series is found.

$$\text{If } r=1,\ \varphi(n)=\frac{4(2n+1)(n^2+n-1)(n^2+n+1)}{(n^2+1)^2[(n+1)^2+1]^2}.$$

Rejecting the factor 4 because it is a square, the series is

$$\frac{3.1.3}{2^2.5^2}+\frac{5.5.7}{5^2.10^2}+\frac{7.11.13}{10^2.17^2}+\frac{9.19.21}{17^2.26^2}+\ldots\ldots+\frac{(2n+1)(n^2+n-1)(n^2+n+1)}{(n^2+1)^2[(n+1)^2+1]^2}+$$

The sum of this series is $\frac{1}{4}$, a square. The sum of n terms is

$$\frac{1}{4}\left(\frac{n^2+2n}{(n+1)^2+1}\right)^2,$$

a square; and the sum of the remaining terms is

$$\left(\frac{n+1}{(n+1)^2+1}\right)^2, \text{ a square.}$$

SOLUTION BY PROF. J. SCHEFFER, COLLEGE OF ST. JAMES, MD.

Putting $a\div(1-q)=x^2$, $a(1-q^n)\div(1-q)=y^2$, $x^2-y^2=z^2$, which equations express the three conditions required, we obtain from the third, putting $y=x-p$,

$$x=\frac{p^2+z^2}{2p},\qquad y=\frac{z^2-p^2}{2p}.$$

Substituting in the second, we get $1-q^n=z^2\div x^2$, whence

$$q=\sqrt[n]{\left(\frac{z}{x}\right)^2}=\left(\frac{2pz}{p^2+z^2}\right)^{\frac{2}{n}},$$

and from the first

$$a=\left(\frac{z^2+p^2}{2p}\right)^2\left[1-\left(\frac{2pz}{p^2+z^2}\right)^{\frac{2}{n}}\right],$$

in which expression z and p can be arbitrarily chosen.

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200. "Prove that the locus of points, which have the sum of the squares of their distances from fixed points constant, is the surface of a sphere."

SOLUTION BY H. HEATON, SABULA, IOWA.

Taking the center of gravity of the fixed points as the origin, let a_1, b_1, c_1; a_2, b_2, c_2, ... a_n, b_n, c_n be their co-ordinates, and x, y, z, the co-ordinates of the points whose locus is required. Then we have

$$(x-a_1)^2+(y-b_1)^2+(z-c_1)^2+\ldots(x-a_n)^2+(y-b_n)^2+(z-c_n)^2=C. \quad (1)$$

$$\therefore\ nx^2+ny^2+nz^2-2(a_1+a_2\ldots a_n)x-2(b_1+b_2\ldots b_n)y-2(c_1+c_2\ldots c_n)z$$
$$+a_1^2+b_1^2+c_1^2\ldots\ldots+a_n^2+b_n^2+c_n^2=C. \quad (2)$$

But since the center of gravity is the origin, $a_1+a_2\ldots a_n$, $b_1+b_2\ldots b_n$ and $c_1+c_2\ldots c_n$, each equals nothing, $\therefore$

$$x^2+y^2+z^2=\frac{1}{n}\Big(C-a_1^2-b_1^2-c_1^2\ldots-a_n^2-b_n^2-c_n^2\Big),$$

the equation of the surface of a sphere.

SOLUTION BY PROF. P. E. CHASE.

The fixed points are at the extremities of a hypothsnuse, $\therefore$ the locus in any possible plane is a circle; and in all possibl planes, a sphere of which the fixed hypothenuse is a diameter.

[Miss Ladd writes, "Question 200 is a well known proposition. It is given by Catalan, *Théorèmes et Problèmes de la Géometrie Elémentaire*, V., 18."]

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201. "Given $\qquad x=a\cos A\pm r_1,$
$\qquad\qquad\qquad y=a\sin A\pm r_2,$

where r_1 and r_2 are the probable errors of x and y. Required the probable errors of a and A."

SOLUTION BY E. L. DE FOREST, WATERTOWN, CONN.

From the approximate relations

$$x=a\cos A \qquad\qquad y=a\sin A$$

we have

$$a^2=x^2+y^2, \qquad\qquad \tan A=y\div x.$$

Actual errors may be regarded as finite differences, and if these are small they may be found in the same way as differentials. Hence,

$$\left.\begin{aligned} a\Delta a&=x\Delta x+y\Delta y \\ \Delta a&=\frac{1}{a}\Big(x\Delta x+y\Delta y\Big) \end{aligned}\right\} \qquad \left.\begin{aligned} \frac{\Delta A}{\cos^2 A}&=\frac{x\Delta y-y\Delta x}{x^2} \\ \Delta A&=\frac{1}{a^2}\Big(x\Delta y-y\Delta x\Big) \end{aligned}\right\}$$

Since Δx and Δy are independent of each other, and their probable values are r_1 and r_2, we have, by a well known rule,

$$(\text{Prob. Error of } a)=\pm\frac{1}{a}\sqrt{\Big(x^2r_1^2+y^2r_2^2\Big)},$$

$$(\text{Prob. Error of } A)=\pm\frac{1}{a^2}\sqrt{\Big(x^2r_2^2+y^2r_1^2\Big)},$$

and these may also be written

$$(\text{P. E. of } a) = \pm \sqrt{[r_1^2 \cos^2 A + r_2^2 \sin^2 A]},$$
$$(\text{P. E. of } A) = \pm \frac{1}{a}\sqrt{\left(r_2^2 \cos^2 A + r_1^2 \sin^2 A\right)}.$$

SOLUTION BY PROF. E. W. HYDE, CINCINNATI, OHIO.

If we have $z = f(z_1, z_2, z_3$ &c.) in which z_1, z_2 &c. are independent observations, and if r, r_1, r_2 &c. are the probable errors of the quantities z, z_1, z_2 &c., then

$$r = \sqrt{\left\{ \left[r_1 \frac{df}{dz_1}\right]^2 + \left[r_2 \frac{df}{dz_2}\right]^2 + \left[r_3 \frac{df}{dz_3}\right]^2 + \&c. \right\}}.$$

From the given relations $x = a \cos A$, $y = a \sin A$ we find

$$a = \sqrt{(x^2+y^2)} \pm r_a, \qquad A = \tan^{-1}\frac{y}{x} \pm r_A;$$

$$\therefore [\text{Prob. er. of } a] = r_a = \left\{ \left(r_1 \frac{da}{dx}\right)^2 + \left(r_2 \frac{da}{dy}\right)^2 \right\}^{\frac{1}{2}} = \left\{ \frac{r_1^2 x^2 + r_2^2 y^2}{x^2+y^2} \right\}^{\frac{1}{2}},$$

$$[\text{Prob. er. of } A] = r_A = \left\{ \left(r_1 \frac{dA}{dx}\right)^2 + \left(r_2 \frac{dA}{dy}\right)^2 \right\}^{\frac{1}{2}} = \left\{ \frac{r_1^2 y^2 + r_2^2 x^2}{x^2+y^2} \right\}^{\frac{1}{2}}.$$

SOLUTION BY CHAS. H. KUMMELL, DETROIT, MICH.

We have $a = \sqrt{[(x \pm r_1)^2 + (y \pm r_2)^2} = \sqrt{(x^2+y^2)} \pm \frac{xr_1 + yr_2}{\sqrt{(x^2+y^2)}} + \ldots.$

$$A = \tan^{-1}\frac{y \pm r_2}{x \pm r_1} = \tan^{-1}\frac{y}{r} \pm \frac{xr_2 - yr_1}{x^2+y^2} + \ldots.$$

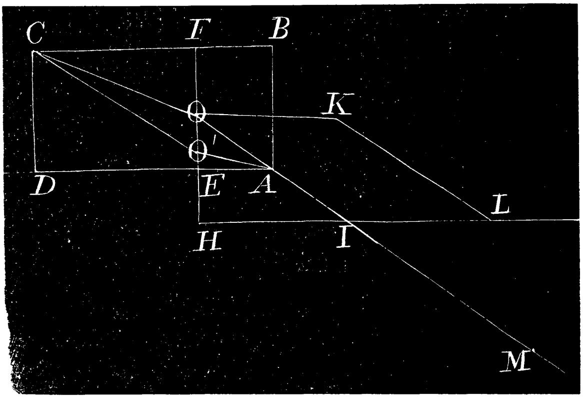
But in applying the method of least squares to non-linear functions, small quantities like probable error, mean error, corrections to observed quantities &c., are considered as differentials, i. e., they are considered small enough to omit their higher powers, therefore (see ANAL., Vol. III, p. 140)

$$r_a = \frac{1}{a}\left(xr_1 + yr_2\right),$$
$$r_A = \frac{1}{a^2}\left(xr_2 - yr_1\right).$$

SOLUTION OF 188 (SEE P. 57) BY P. RICHARDSON, BROOKLYN, N. Y.

Let AOC be the route of quickest travel from A to C, M be any point in OA produced, then is $MAOC$ the route of quickest travel from M to C.

Let $AO'C$ be any other route from A to C; then the time in $AOC = AO \div 2m + OC \div m$ and the time in $AO'C = AO' \div 2m + O'C \div m$. By hypothesis $AO \div 2m + OC \div m < AO' \div 2m + O'C \div m$, and if we add the time in MA ($MA \div 2m$) to both sides of this inequality the inequality will still exist; $\therefore$ if AOC is the route of quickest travel from A to C, $MAOC$ is the route of quickest travel from M to C.



Take any point O in EF. Draw $OI = OC$, and through I draw HL perp. to OE produced. Take $IM = OC$ and draw OK and KL each equal to OC. The rate of travel may be regarded as 2 and 1.

The time in KOC at rate $1 = OC$ at rate $1+$ MO at rate 2, and time from K to C at rate 1 will be least when KOC is a straight line, and when it is least its half will be least likewise. The time from L to O at rate 2 will be least when LKO is a straight line, and at rate 2 it is equal half the time in KOC at rate 1. KO is common to KOC and LKO, $\therefore$ $LKOC$ must be a straight line; $\therefore$ because $LO = 2OC$, $HO = 2OF$.

In the triangles OIH and OAE we have $OI : OH :: OA : OE$. Substituting CO for OI and 2OF for OH and multiplying extremes and means,

$$CO\times OE = 2OF\times OA. \qquad (1)$$

Putting $AE = a$, $CF = b$, $EF = c$, $EO = x$ and $FO = c-x$, we get

$$x\sqrt{[b^2+(c-x)^2]} = 2(c-x)\sqrt{[x^2+a^2]}.$$

QUERY. “Which of the two following statemens is correct?

‘The point $x'y'z'$ is a multiple point of the order k, if all the differential coefficients of the order $k-1$ vanish for that point.” (Salmon, *Higher Plane Curves*, Art. 73.)

‘Il s'ensuit que, pour un point multiple de l'ordre n, il est *nécessaire, mais non suffisant*, que les dérivées partielles de la fonction f des ordres inférieurs à n s'annulent tontes pour les co-ordonnées du point m.’ (Serret, *Calcul. Différential*, Art. 16.)”

[Chas. H. Kummell writes, “Mr. Serret evidently means *real* multiple points, viz., the case where the equation of the nth degree for $\frac{dy}{dx}$ has more than one real root, in which case we have a *visible* multiple point.”

A. S. Hathaway writes, “Salmon and Serret differ as to what is the order of a multiple point whose coordinates x_0, y_0 make the partial derivatives of f, below the nth order vanish.

Serret reckons by the number of *real* branches at the point. * * *

Salmon means by a multiple point of order n, a point, every line through which meets the curve there in n coincident points.” * * *